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Scale-by-scale budget and similarity laws for shear turbulence

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Turbulent shear flows, such as those occurring in the wall region of turbulent boundary layers, show a substantial increase of intermittency in comparison with isotropic conditions. This suggests a close link between anisotropy and intermittency. However, a rigorous statistical description of anisotropic flows is, in most cases, hampered by the inhomogeneity of the field. This difficulty is absent for homogeneous shear flow. For this flow the scale-by-scale budget is discussed here by using the appropriate form of the Kármán–Howarth equation, to determine the range of scales where the shear is dominant. The resulting generalization of the four-fifths law is then used to extend to shear-dominated flows the Kolmogorov–Oboukhov theory of intermittency. The procedure leads naturally to the formulation of generalized structure functions, and the description of intermittency thus obtained reduces to the K62 theory for vanishing shear. The intermittency corrections to the scaling exponents are related to the moments of the coarse-grained energy dissipation field. Numerical experiments give indications that the dissipation field is statistically unaffected by the shear, supporting the conjecture that the intermittency corrections are universal. This observation together with the present reformulation of the theory gives a reason for the increased intermittency observed in the classical longitudinal velocity increments.

1. Introduction

At large Reynolds number, turbulent flows exhibit strong intermittent effects which appear in a number of ways, such as intense coherent structures, anomalous scaling in the inertial range, and exponential tails in the probability density functions (see e.g. Frisch 1995). In the case of homogeneous and isotropic turbulence, more than twenty years of intense scientific research provides us with strong indications that intermittency shows universal properties (Sreenivasan & Antonia 1997).

For inhomogeneous anisotropic turbulence, a well-established conclusion concerning universal properties of intermittency is lacking, although some preliminary results have been proposed (see for instance Toschi, Leveque & Ruiz-Chavarria 2000; Danaila *et al.* 2001; Arad *et al.* 1999; Kurien & Sreenivasan 2001; Biferale & Vergassola 2001; Schumacher, Sreenivasan & Yeung 2003). A particular case of anisotropic turbulence is homogeneous shear flows, which have been the subject of a number of numerical (see Pumir 1996 and references therein) and experimental investigations (Shen & Warhaft 2002 and references therein). Homogeneous shear flows are probablythe simplest non-trivial example of anisotropic turbulence where issues such as the generation and the dynamics of coherent vortical structures, the effect of the large-scale anisotropic forcing on the small-scale intermittent fluctuations of velocity and the universal (if any) scaling properties of velocity increments in the inertial range can be addressed.

In the present paper we investigate intermittency effects in homogeneous shear flows by using a long-time highly resolved direct numerical simulation (DNS), see Gualtieri *et al.* (2002*a*) for details. The integration time is $ST_{max} = 5900$ while the Taylor Reynolds number and the shear parameter are $Re_\lambda \simeq 45$ and $S^* = Sq^2/\bar{\epsilon} \simeq 7$ respectively. We denote by S the mean shear, $q^2/2$ the turbulent kinetic energy and $\bar{\epsilon}$ the energy dissipation rate. For this simulation we have $l_d/L_s \simeq 20$ and $L_s/\eta \simeq 20$ with $L_s = \sqrt{\bar{\epsilon}/S^3}$ the shear scale, l_d and η the integral and the dissipative scales respectively.

Our basic idea is to study the fluctuations of the velocity field by using a generalization of the Kolmogorov four-fifths equation. Although the isotropic constraints cannot be for homogeneous shear flow, we are still able to write the appropriate form of this equation following Hinze (1959). An advantage is that we can immediately isolate the contributions due to the mean shear to identify the range of scales where they become the leading terms. Next, by generalizing the concept of structure functions, we are able to study systematically the scaling properties of shear turbulence and its universality with respect to homogeneous and isotropic conditions.

In order to clarify the analogy between the Kolmogorov–Oboukhov approach and the present formulation, the classical theory is briefly summarized here. The Kolmogorov equation for homogeneous isotropic turbulence follows from the Kármán–Howarth equation by re-expressing the correlation function in terms of velocity differences,

$$\langle \delta V_\parallel^3 \rangle = -\tfrac{4}{5}\bar{\epsilon} r + 6\nu \frac{\mathrm{d}}{\mathrm{d}r}\langle \delta V_\parallel^2 \rangle, \tag{1.1}$$

where $\delta V_\parallel = [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})]\cdot \boldsymbol{r}/r$ are the longitudinal velocity increments. In the inertial range, this equation reduces to the well-known four-fifths law relating the third-order structure function to the mean rate of energy dissipation $\bar{\epsilon}$ and to the separation r. In the absence of intermittency effects the four-fifths law suggests that, in addition to the separation r, any inertial-range scalings should involve only the mean rate of energy dissipation $\bar{\epsilon}$, or, more properly, the average energy flux through the inertial range, leading to dimensional scaling for structure functions of order p. When intermittency is accounted for, scaling laws are nonetheless observed in the inertial range, with exponents substantially different from the dimensional prediction for large p (Benzi *et al.* 1995 and references therein).

In the context of the Kolmogorov refined similarity hypothesis (RKSH) (Kolmogorov 1962) the intermittency corrections are linked to the statistical properties of the coarse-grained energy dissipation,

$$\epsilon_r = \frac{1}{B_r}\int_{B_r} \epsilon_{loc}\,\mathrm{d}V_r, \tag{1.2}$$

with ϵ_{loc} the local rate of turbulent kinetic energy dissipation, by expressing the structure functions as

$$\langle \delta V_\parallel^p \rangle \propto \langle \epsilon_r^{p/3} \rangle r^{p/3}, \tag{1.3}$$

i.e. the intermittency of the velocity field is originated by an intermittent energy cascade which ends up as an intermittent dissipation field. The link between the two fields is provided by (1.3) where $\langle \epsilon_r^{p/3} \rangle \propto r^{\tau(p/3)}$. Intermittency effects on velocity structure functions in the inertial range have been established beyond any doubt (see e.g. Zhou & Antonia 2000). More controversial is the assessment of (1.3). However, the evidence is all in favour of this equation, which, though not originally derived from first principles, describes correctly the physics of small-scale intermittency (Wang *et al.* 1996). In this context, the four-fifths law is an exact constraint to be fulfilled by any acceptable theory of turbulence. Also, the Kolmogorov and the Kármán–Howarth equations may also suggest the basic ingredients of a reasonable turbulence theory.

We can now re-express more technically the aim of the present paper: starting from the extension of the Kármán–Howarth equation to homogeneous shear flow (see also Oberlack 2001 for a related derivation, and Hill 2001 for the equations for high-order structure functions in general inhomogeneous flows), we are interested in generalizing the RKSH to homogeneous anisotropic turbulence. As we shall see, the proposed similarity law includes the classical Kolmogorov–Oboukhov theory of intermittency as a special case. We intend to check the proposed form of similarity by using DNS results for a homogeneous shear flow. This may complicate matters, since, given the limitations of DNS, scaling laws cannot be detected in terms of separation. To overcome the difficulty we analyse our data via extended self-similarity (ESS) (Benzi *et al.* 1995), to show that, indeed, the generalized RKSH holds for homogeneous shear flow. Actually ESS scalings, $\langle \delta V_\parallel^p \rangle \propto \langle \delta V_\parallel^q \rangle^{\zeta(p|q)}$ with $\zeta(p|q) \neq p/q$, are also observed at moderate Reynolds number both for isotropic and for shear turbulence (see Antonia, Zhou & Romano 2002, Ruiz Chavarria *et al.* 2000 and Toschi *et al.* 1999). In particular, for homogeneous isotropic turbulence, in the spirit of ESS, one can rewrite (1.3) as (Benzi *et al.* 1996)

$$\left\langle \delta V_\parallel^p \right\rangle \propto \left\langle \epsilon_r^{p/3} \right\rangle / \bar{\epsilon}^{p/3} \left\langle \delta V_\parallel^3 \right\rangle^{p/3}. \tag{1.4}$$

The generalization of the Kolmogorov–Oboukhov theory proposed here rationalizes a number of recent results concerning intermittency in shear-dominated turbulence. A number of papers have recently appeared on this subject. Strictly related to the present results is Benzi *et al.* (1999) where a new form of similarity law has been proposed to explain the increased intermittency of the near-wall region of wall-bounded flows noticed by a number of authors (see Ruiz Chavarria *et al.* 2000; Toschi *et al.* 1999). That form of similarity law has been checked using experimental (Jacob, Olivieri & Casciola 2002) and DNS data (Gualtieri *et al.* 2002*a*), to finally address the possible coexistence of two different intermittent regimes (Casciola *et al.* 2002). In the present paper we discuss a unifying formulation encompassing both the classical RKSH and the form proposed by Benzi *et al.* (1999).

Although the Kármán–Howarth equation for homogeneous isotropic turbulence is conveniently expressed in terms of third- and second-order longitudinal structure functions, more generally it involves integrals on a sphere of radius r, see Nie & Tanveer (1999), such as

$$\oint_{\partial B_r} \langle \delta V^2 \delta V_i \rangle n_i \, \mathrm{d}S_r \tag{1.5}$$

etc. The primary form of the Kármán–Howarth equation is recast in the more familiar form (1.1) by using isotropy. Clearly, the last step cannot be applied to anisotropic turbulence. The use of sphere averages, such as (1.5), allows one to write an equation which directly involves the coarse-grained dissipation field of the Kolmogorov–Oboukhov theory, Gualtieri *et al.* (2002*b*) (see also Hill 2002 and Lindborg 1996 for a related discussion). Consistently the generalized Kármán–Howarth equation for homogeneous shear flow gives a relation between integrals on the sphere, implying that the generalized RKSH will also maintain the same flavour.

2. The Kármán–Howarth equation

In terms of fluctuating components, $\boldsymbol{u}$, a homogeneous field $\boldsymbol{v}$ may be written as

$$v_i = u_i + \frac{\partial U_i}{\partial x_k} x_k. \tag{2.1}$$

Here the velocity field obeys the incompressible Navier–Stokes equations and the mean component is solenoidal, $\partial U_i/\partial x_i = 0$. Standard notation is used for quantities evaluated at $\boldsymbol{x}$ and a prime denotes those at $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{r}$. Multiplying the equation for u_i by u'_j and that for u'_j by u_i, adding the results and averaging we obtain the equation for the two-point correlation tensor $R_{i,j}(\boldsymbol{r}, t) = \langle u_i u'_j \rangle$, where the angular brackets denote an ensemble average. The equation for the trace $R_{i,i}$ is

$$\frac{\partial R_{i,i}}{\partial t} + \frac{\partial}{\partial r_k}[\langle u_i u'_i u'_k\rangle - \langle u_i u'_i u_k\rangle] + (U'_k - U_k)\frac{\partial R_{i,i}}{\partial r_k} = -\frac{\partial U_i}{\partial x_k}(R_{i,k} + R_{k,i}) + 2\nu \frac{\partial^2 R_{i,i}}{\partial r_k \partial r_k}, \tag{2.2}$$

where, by homogeneity, $\partial/\partial x_k \langle \bullet \rangle = -\partial/\partial r_k \langle \bullet \rangle$ and $\partial/\partial y_k \langle \bullet \rangle = \partial/\partial r_k \langle \bullet \rangle$, and the pressure term disappears due to incompressibility. Equation (2.2) can be re-expressed in terms of structure functions, by considering the velocity increments

$$\delta V_i(\boldsymbol{x}, \boldsymbol{r}) = u_i(\boldsymbol{x} + \boldsymbol{r}) - u_i(\boldsymbol{x}), \tag{2.3}$$

$\delta V^2 = \delta V_i \delta V_i$ and $\delta U_k = U'_k - U_k$. Since $R_{i,i} = \langle u_i u_i \rangle - \frac{1}{2}\langle \delta V^2 \rangle$ and $R_{i,k} + R_{k,i} = 2\langle u_i u_k \rangle - \langle \delta V_i \delta V_k \rangle$, and considering the standard manipulations

$$\langle \delta V^2 \delta V_k \rangle = -2[\langle u_i u'_i u'_k \rangle - \langle u_i u'_i u_k \rangle] + \langle u_i u_i u'_k \rangle - \langle u'_i u'_i u_k \rangle, \tag{2.4}$$

the equation for the velocity increments, in steady-state conditions, follows as

$$\frac{\partial}{\partial r_k}\langle \delta V^2 \delta V_k + \delta V^2 \delta U_k \rangle + 2\frac{\partial U_i}{\partial x_k}\langle \delta V_i \delta V_k \rangle = 4\frac{\partial U_i}{\partial x_k}\langle u_i u_k \rangle + 2\nu \frac{\partial^2}{\partial r_k \partial r_k}\langle \delta V^2 \rangle, \tag{2.5}$$

where use has been made of incompressibility. Here the first term on the right-hand side is identified with the average dissipation rate, given the balance between production of turbulent kinetic energy and viscous dissipation which holds for homogeneous flows,

$$\frac{\partial U_i}{\partial x_k}\langle u_i u_k \rangle = -\nu \left\langle \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k} \right\rangle = -\bar{\epsilon}\,. \tag{2.6}$$

All averaged terms in (2.5) are only a function of the separation vector and integration over a sphere of radius r yields

$$\begin{aligned}\oint_{\partial B_r} \langle \delta V^2 \delta V_i \rangle n_i + \langle \delta V^2 \delta U_i \rangle n_i \, \mathrm{d}S_r + \int_{B_r} 2\frac{\partial U_i}{\partial x_k}\langle \delta V_i \delta V_k \rangle \, \mathrm{d}V_r \\ = -4\bar{\epsilon} r + 2\nu r^2 \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^2}\oint_{\partial B_r}\langle \delta V^2 \rangle \, \mathrm{d}S_r\right).\end{aligned} \tag{2.7}$$

We stress that, for a homogeneous shear flow, (2.7) follows exactly from the Navier–Stokes equation, with no approximation whatsoever. Following the procedure described by Hill (2002) a more general form of this equation can be derived, allowing the assumption of homogeneity to be removed. It reduces to the present form (2.7) under the assumption of full homogeneity, as considered here.

Selecting the coordinate axis so that $\boldsymbol{U} = (Sy,\, 0,\, 0)$ and by denoting by n_1 and n_2 the components of the normal to the sphere in the directions of mean flow and mean gradient, respectively, (2.7) specializes as

$$\frac{1}{4\pi r^2}\oint_{\partial B_r} \langle \delta V^2 \delta V_i \rangle n_i \; + n_1 n_2 S r \langle \delta V^2 \rangle \,\mathrm{d}S_r + \frac{1}{4\pi r^2}\int_{B_r} 2S\langle \delta u_1 \delta u_2 \rangle \,\mathrm{d}V_r$$
$$= -\tfrac{4}{3}\bar{\epsilon} r + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{4\pi r^2}\oint_{\partial B_r} \langle \delta V^2 \rangle \,\mathrm{d}S_r \right). \tag{2.8}$$

This equation may be interpreted as a scale-by-scale budget in physical space, as becomes clear by considering the spectral form of the Kármán–Howarth equation, (2.2) (see e.g. Townsend 1956).

3. Scale-by-scale budget

In homogeneous isotropic turbulence the Kármán–Howarth equation describes the process of energy cascade towards small scales and establishes the balance between energy flux and dissipation according to the four-fifths law (1.1). The energy, injected at the integral scale, is simply transferred across the inertial range to be dissipated at scales of the order of the Kolmogorov length η. Equation (2.8) extends this result to homogeneous anisotropic flows where the cascade process is strongly modified by the continuous energy injection associated with the mean velocity gradient.

Following the derivation of the previous section, the nonlinear terms generate two distinct contributions. In particular, the correlation

$$S_3^{tr} = \frac{1}{4\pi r^2}\oint_{\partial B_r} \langle \delta V^2 \delta V_i \rangle n_i + n_1 n_2 S r \langle \delta V^2 \rangle \,\mathrm{d}S_r, \tag{3.1}$$

where the superscript tr stays for transfer, is the proper generalization of the third-order structure function $\langle \delta V_{\parallel}^3 \rangle$ to anisotropic flows. Its meaning is clear: the flux across the surface ∂B_r is due to transport of turbulent kinetic energy by both the mean flow and by the turbulent fluctuations. It arises from the terms in divergence form in (2.5), as appropriate for a flux. The second term on the left-hand side of (2.8),

$$S_3^{pr} = \frac{1}{4\pi r^2}\int_{B_r} 2S\langle \delta u_1 \delta u_2 \rangle \,\mathrm{d}V_r, \tag{3.2}$$

where the subscript pr stands for production, describes the fluctuations in the rate of turbulent kinetic energy production up to the scale r.

The budget established by (2.8) is shown in figure 1 where the ensemble averages implied by the angular brackets are replaced by space–time averages. Within the accuracy of the available statistics, the different terms sum to zero for all separations r.

In figure 1(b), transfer S_3^{tr} and production S_3^{pr} are compared to $-(4/3)\bar{\epsilon} r$ minus the finite Reynolds number contributions. At the large scales $\bar{\epsilon}$ is balanced by production. At small scales $\bar{\epsilon}$ is instead balanced by energy transfer. On decreasing towards the scales of the classical inertial subrange where production becomes ineffective, the energy injected at the larger scales piles up and originates the flux of energy which, transferred throughout the cascade, is eventually dissipated. In fact the production and the transfer terms achieve the same order of magnitude at the characteristic length scale where $S_3^{tr} \simeq S_3^{pr}$. The cross-over scale may be estimated on dimensional grounds

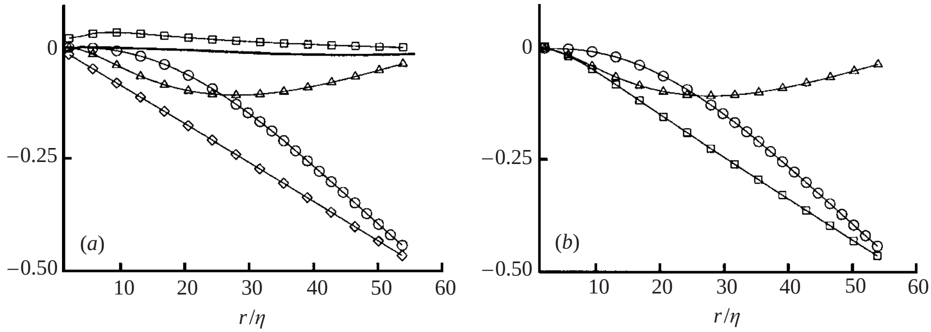

FIGURE 1. (*a*) The terms of (2.8) vs. r/η: S_3^{tr} ($\triangle$), S_3^{pr} ($\bigcirc$), the viscous correction ($\square$), and $-\frac{4}{3}\bar{\epsilon}r$ ($\Diamond$). The four terms sum to zero within the accuracy of the available statistics, solid line. (*b*) $\square$, $-\frac{4}{3}\bar{\epsilon}r + r^2\mathrm{d}/\mathrm{d}r(1/r^2 \oint_{\partial B_r} 2\nu\langle\delta V^2\rangle\,\mathrm{d}S_r)$, other symbols defined as for (*a*).

by assuming $S_3^{tr} \simeq \bar{\epsilon}r$ and $S_3^{pr} \simeq Sr\bar{\epsilon}^{2/3}r^{2/3}$ where the last estimate follows from the fact that, dimensionally, S_3^{pr} is given by $Sr\delta u_1\delta u_2$, and the velocity increments are order $\bar{\epsilon}^{1/3}r^{1/3}$. It follows that $L_s = \sqrt{\bar{\epsilon}/S^3}$. This is actually well-supported by the data shown in figure 1 where the balance is achieved at $r_b/\eta \simeq 25$, roughly corresponding to the dimensional prediction of $L_s/\eta \simeq 20$.

4. Similarity laws

In the extended inertial range (i.e. including the production range above L_s), the asymptotics of (2.8), in terms of the generalized structure function $S_3 = S_3^{tr} + S_3^{pr}$, is

$$S_3 \propto \bar{\epsilon}r. \tag{4.1}$$

This equation represents the shear flow analogy of the four-fifths law of homogeneous isotropic turbulence, to which it nicely reduces for zero shear.

Broadly speaking, the Kolmogorov–Oboukhov description of intermittency, (1.3), amounts to the statistical equivalence of two random processes, namely the coarse-grained dissipation field, (1.2), and the third power of the longitudinal velocity increment. To extend this theory to homogeneous shear flow, we can keep the coarse-grained field, ϵ_r, of the turbulent kinetic energy dissipation ϵ_{loc}, as the descriptor of intermittency corrections. However, (2.8) inhibits the use of the longitudinal velocity increments. To extend the generalized third-order structure function S_3 to higher orders,

$$\tilde{S}_p = \langle\widetilde{\delta V}^p\rangle, \tag{4.2}$$

we may suitably define extended velocity increments as

$$\widetilde{\delta V} = \left| \frac{1}{4\pi r^2} \oint_{\partial B_r} \delta V^2 \delta V_i\, n_i\; + n_1 n_2\, Sr\, \delta V^2\, \mathrm{d}S_r + \frac{2S}{4\pi r^2} \int_{B_r} \delta u_1 \delta u_2\, \mathrm{d}V_r \right|^{1/3}. \tag{4.3}$$

A qualitatively similar combination of terms, though empirically involving longitudinal structure functions of second and third order combined with a suitable tuning constant, may be found in the integral structure functions used in Toschi *et al.* (2000).

On purely dimensional grounds, (4.1) is expected to generalize to $\tilde{S}_p \propto \bar{\epsilon}^{p/3}r^{p/3}$, corresponding to the extension to the homogeneous shear flow of Kolmogorov's non-intermittent 1941 formulation. Intermittency corrections can then be included

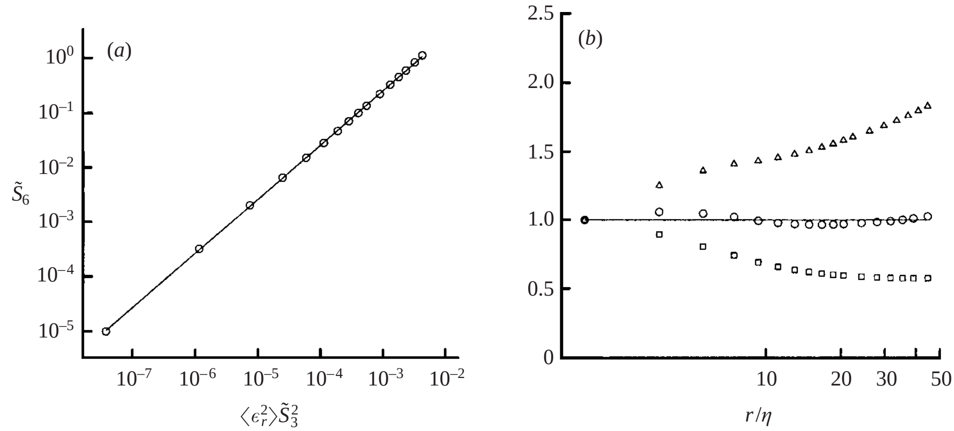

FIGURE 2. (*a*) $\tilde{S}_6$ vs. $\langle\epsilon_r^2\rangle\tilde{S}_3^2$ (○). Data are fitted in the whole range of scales by a power law with slope $s = 1$ (solid line). The shear scale approximately corresponds to the sixth point from the left. (*b*) The ratio $\tilde{S}_6/(\langle\epsilon_r^2\rangle\tilde{S}_3^2)^\alpha$ vs. separation r/η for $\alpha = 1.05$ (△), $\alpha = 1.00$ (○), $\alpha = 0.95$ (□).

following the same line of reasoning that led to (1.3), to obtain

$$\tilde{S}_p \propto \left\langle \epsilon_r^{p/3} \right\rangle r^{p/3}. \tag{4.4}$$

Unfortunately we cannot validate equation (4.4) directly, since, given the limitations on the Reynolds number of our DNS, no power-law behaviour can be observed when using the separation r as scaling variable. However, the concept of extended self-similarity can help overcoming the difficulty by recasting our ansatz in the form

$$\tilde{S}_p \propto \left\langle \epsilon_r^{p/3} \right\rangle / \bar{\epsilon}^{p/3} \tilde{S}_3^{p/3}. \tag{4.5}$$

As shown in figure 2 for $p = 6$, the data we have available support equation (4.5), which apparently holds uniformly for all scales. This is a significant feature of the present description, where both the production and the transfer term are retained and the two contributions are evaluated exactly avoiding any approximation.

As a further check, we also report in figure 2(*b*) the ratio $\tilde{S}_6/(\langle\epsilon_r^2\rangle\tilde{S}_3^2)^\alpha$ for different αs. The value $\alpha = 1$, giving (4.5), achieves the optimal compensation around a constant for the whole range of scales. Quantitatively, the compensated data for (4.5) fluctuate about 4.5% giving clear evidence in favour of the proposed similarity law above and below the shear scale L_s.

According to (4.5), the random function $\delta\tilde{V}^3$ is statistically equivalent to the coarse-grained energy dissipation ϵ_r. For the present data where the shear affects the turbulent fluctuations sufficiently to break the classical refined similarity, the statistics of the field of turbulent kinetic energy dissipation rate show no appreciable difference with homogeneous isotropic turbulence, as shown in Gualtieri *et al.* (2002*a*). There ϵ_r was evaluated by using one-dimensional integrals. Here we have strictly used (1.2), with no significant difference in the results discussed in the previous paper.

The above observation and (4.5) allow the conclusion that the generalized structure functions $\tilde{S}_p$ are unaffected by the shear, see figure 3(*a*). Actually, the relative scaling exponents of $\tilde{S}_p$ vs. $\tilde{S}_3$ are, within 1% accuracy, the same as found in isotropic conditions. In this respect, they extend to shear-dominated flows the universal behaviour well-documented in homogeneous isotropic turbulence.

We also compare the present findings with previous related work concerning the similarity law proposed by Benzi *et al.* (1999) for shear-dominated flows and

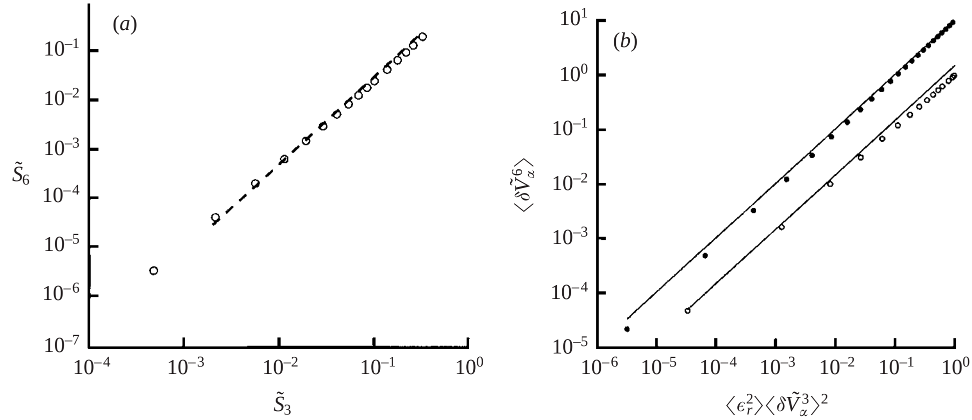

FIGURE 3. (a) $\tilde{S}_6$ vs. $\tilde{S}_3$ in homogeneous isotropic turbulence (dashed line, slope $s = 1.78$) and in the homogeneous shear flow (○). (b) $\langle\delta\tilde{V}_\alpha^6\rangle$ vs. $\langle\epsilon_r^2\rangle\langle\delta\tilde{V}_\alpha^3\rangle^2$. Production ($\alpha = pr$) (●), transfer ($\alpha = tr$) (○). The slope of the two solid lines is $s = 1$.

the double intermittent behaviour in shear turbulence discussed by Casciola *et al.* (2002). For $r < L_s$, the budget provided by the Kármán–Howarth equation suggests that $|\tilde{S}_3^{tr}| \gg |\tilde{S}_3^{pr}|$. In this limit (4.5) reduces to the classical Kolmogorov–Oboukhov similarity,

$$\left\langle\delta\tilde{V}_{tr}^{p}\right\rangle \propto \left\langle\epsilon_r^{p/3}\right\rangle\left\langle\delta\tilde{V}_{tr}^{3}\right\rangle^{p/3}, \tag{4.6}$$

where the subscript tr indicates that only the terms in divergence form related to inertial transfer are retained in expression (4.3) for $\delta\tilde{V}$. This equation, plotted for $p = 6$ on figure 3(b), is shown to hold below the shear scale, where the open symbols approach the solid line. Equation (4.6) still involves integrals over a sphere. In view of the comparison, it may be recast into the familiar form involving only the longitudinal increments, evaluated in the streamwise direction δu_1. This is done by assuming that, as in the homogeneous isotropic case, $\delta\tilde{V}_{tr}$ behaves statistically as $A(r)\delta u_1$ where $A(r)$ is a suitable non-fluctuating function. Hence (4.6) is re-written as $\langle\delta u_1^p\rangle \propto \langle\epsilon_r^{p/3}\rangle\langle\delta u_1^3\rangle^{p/3}$. On the other hand in the range of scales $r > L_s$ the scale-by-scale budget shows that the only productions terms in (4.5) are relevant. In this range we obtain asymptotically

$$\left\langle\delta\tilde{V}_{pr}^{p}\right\rangle \propto \left\langle\epsilon_r^{p/3}\right\rangle\left\langle\delta\tilde{V}_{pr}^{3}\right\rangle^{p/3}, \tag{4.7}$$

where the subscript pr denotes that only the volume integral term related to turbulent kinetic energy production has been retained in the definition of $\delta\tilde{V}$. Equation (4.7) is also plotted for $p = 6$ on figure 3(b) and clearly holds at the largest scales, where the filled symbols approach the solid line. Equation (4.7) corresponds to the similarity law originally proposed by Benzi *et al.* (1999). Following analogous arguments to those for $\delta\tilde{V}_{tr}^3$, we may estimate $\delta\tilde{V}_{pr}^3$ in terms of $B(r)\delta u_1\delta u_2$, where $B(r)$ is, again, a non-fluctuating function. Hence, by assuming the same statistical behaviour for $\delta u_1\delta u_2$ and δu_1^2 as in Benzi *et al.* (1999) and Toschi *et al.* (2000), the similarity law $\langle\delta u_1^p\rangle \propto \langle\epsilon_r^{p/2}\rangle\langle\delta u_1^2\rangle^{p/2}$ follows. As discussed in the introduction, this equation has been repeatedly verified, both numerically and experimentally, in a number of different shear flows and has been shown able to capture the change in the intermittency of longitudinal increments across the shear scale.

5. Final remarks

The exact Kármán–Howarth equation for homogeneous shear flow has been used to provide a rational framework for the analysis of intermittency in shear turbulence, i.e. in flows characterized by large-scale anisotropies. Besides confirming that production of turbulent kinetic energy prevails at large scales while transfer is crucial at the smallest ones and that the two ranges are separated by the shear scale L_s, the approach has proven fruitful in allowing the generalization of the Kolmogorov–Oboukhov theory of intermittency to shear-dominated turbulence. In particular the two main achievements of the paper are summarized as follows. A generalization of the structure functions can be introduced in shear turbulence, to extend the classical description in terms of velocity increments used in the standard theory for isotropic flows. The extended structure functions obey a similarity law which, formally, keeps the structure of the Kolmogorov–Oboukhov similarity, in such a way that in the limit of zero shear the classical results of isotropic turbulence are recovered. This generalized similarity law has been favourably checked against DNS data of a homogeneous shear flow and has been shown to be consistent with both the classical refined similarity at small scale and the new form of similarity law recently introduced for shear-dominated turbulence (Benzi *et al.* 1999). While scaling laws in terms of traditional longitudinal structure functions display two different forms of intermittency across the shear scale (Casciola *et al.* 2000), the generalized similarity described in this paper holds uniformly across the entire range of available scales.

A previous investigation, whose results are entirely confirmed by the present analysis, has shown that the properties of the turbulent kinetic energy dissipation related to the description of intermittency of velocity fluctuations are unaltered by the shear, i.e. they are identical to those of isotropic turbulence. On the basis of the present formulation, universality in the intermittency of the dissipation field entails universality of the extended increments of the velocity fluctuations we have defined. By exploiting the relationship between the extended and the standard velocity fluctuation increments we find that the universal intermittency corrections for the extended increments give a reason for the increased intermittency of the standard increments, in agreement with previous numerical and experimental results on shear turbulence. This is due to the correspondence between the pth moment of the longitudinal increments and the $3/2\,p$th extended structure functions, $\tilde{S}_p$, for $r > L_s$.

In conclusion the present description provides an appropriate estimate for the energy flux in anisotropic conditions, a crucial point in developing working closure in LES since the now classical model of Smagorinski. We finally stress the role of homogeneous shear flow as the proper setting to check possible alternative formulations of LES closures.

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